Optiver's Signal Problem

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1 Problem Formulation

Calvin has to cross several signals when he walks from his home to school. Each of these signals operate independently. They alternate every 80 seconds between green light and red light.At each signal, there is a counter display that tells him how long it will be before the current signal light changes. Calvin has a magic wand which lets him turn a signal from red to green instantaneously. However, this wand comes with limited battery life, so he can use it only for a specified number of times.

What is the expected waiting time if he uses the wand optimally?

2 Mathematical Setup

Consider the random variables $T_{s,w}^{x(s,w)}$ for any $x \in (0, 80)$ and $s, w \in \mathbb{N}_0$ representing the waiting time when there are s many signals and he has w many wands which he uses when the number on the counter is higher than $x(s, w)$. We introduce the random variable $C_s \sim \text{Ber}(1/2)$ modelling the colour of the signal s when Calvin arrives at it ($C_s = 0$ means the singal is green). Moreover, the random variable $T_s \sim \text{Unif}(0, 80)$ models the number display by the counter of signal s. Then by the law of total expectation we obtain

$$
\begin{aligned}
&\mathbb{E}[T_{s,w}^{x(s,w)}] \\
=&\mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 1]\mathbb{P}(C_s = 1) + \mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 0]\mathbb{P}(C_s = 0) \\
=&\frac{1}{2}\mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 1] + \frac{1}{2}\mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 0] \\
=&\frac{1}{2}\Bigg(\mathbb{E}[T_{s-1,w}^{x(s,w)}] + \mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 1, T_s > x(s, w)]\mathbb{P}(T_s > x(s, w)) \\
&\qquad + \mathbb{E}[T_{s,w}^{x} \mid C_s = 1, T_s \leq x(s, w)]\mathbb{P}(T_s \leq x(s, w))\Bigg).
\end{aligned}
$$

Clearly, it holds $\mathbb{P}(T_s \leq x(s, w)) = \frac{x(s,w)}{80}$. Moreover, it makes sense that it is optimal to choose the threshold $x(s, w)$ a

$$x(s, w) := \mathbb{E}[T_{s-1,w-1}^{x(s-1,w-1)}] - \mathbb{E}[T_{s-1,w}^{x(s-1,w)}].$$

The rational behind this is, that I am willing to use the wand if the counter is higher than the difference between the expected time I will have to wait in the future if I do not use the wand

and the expected time if I do so. Therefore, we finally obtain the following recursive structure

$$\begin{aligned}
&\mathbb{E}[T_{s,w}^{x(s,w)}] \\
=&\frac{1}{2}\mathbb{E}[T_{s-1,w}^{x(s,w)}] + \frac{x(s,w)}{160}\mathbb{E}[T_{s,w}^{x} \mid C_s = 1, T_s \leq x(s,w)] + \frac{80 - x(s,w)}{160}\mathbb{E}[T_{s,w}^{x(s,w)} \mid C_s = 1, T_s > x(s,w)] \\
=&\frac{1}{2}\mathbb{E}[T_{s-1,w}^{x(s-1,w)}] + \frac{x(s,w)}{160}\left(\frac{x(s,w)}{2} + \mathbb{E}[T_{s-1,w}^{x(s-1,w)}]\right) + \frac{80 - x(s,w)}{160}\mathbb{E}[T_{s-1,w-1}^{x(s-1,w-1)}].
\end{aligned}$$

It is worth noting that the following trivial cases (initial conditions) exist: it holds $\mathbb{E}[T_{s,w}^{x}] = 0$ if $s \leq w$ and $\mathbb{E}[T_{s,0}^{x(s,0)}] = 20s$.

3 Solution for $s = 2$ and $w = 1$

In this case, it follows that $x(2,1) = 20$. Therefore, the recursive formula for expected waiting time reads

$$\mathbb{E}[T_{2,1}^{20}] = \frac{1}{2}\mathbb{E}[T_{1,1}^{20}] + \frac{20}{160}\left(\frac{20}{2} + \mathbb{E}[T_{1,1}^{20}]\right) + \frac{80 - x}{160}\mathbb{E}[T_{1,0}^{20}] = \frac{20^2}{320} + \frac{20(80 - x)}{160} = 8.75.$$

4 Solution for general s and w

You can find the corresponding code at Github.